

# The Multilevel Finite Element Discretizations Based on Local Defect-Correction for Nonsymmetric Eigenvalue Problems

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**Abstract :** Based on the work of Xu and Zhou [Math.Comput., 69(2000), pp. 881-909], we establish new three-level and multilevel finite element discretizations by local defect-correction technique. Theoretical analysis and numerical experiments show that the schemes are simple and easy to carry out, and can be used to solve singular nonsymmetric eigenvalue problems efficiently. We also discuss the local error estimates of finite element approximations; it's a new feature here that the estimates apply to the local domains containing corner points.

**Keywords :** nonsymmetric eigenvalue problems, finite element, multilevel discretization, local refinement, local defect-correction.

**1991 MSC :** code 65N25, 65N30

## 1 Introduction

Nonsymmetric elliptic eigenvalue problems have important physical background, such as convection-diffusion in fluid mechanics, environmental problems and so on. Thus, finite element methods for solving this problem become an important topic which has attracted the attention of mathematical and physical fields: [2] discussed a priori error estimates, [5, 11, 9, 12, 15, 24] a posteriori error estimates and adaptive algorithms, [18] function value recovery algorithms, [16, 25] two level algorithms, [17, 26] extrapolation methods, [5] an adaptive homotopy approach, etc. This paper turns to discuss finite element multilevel discretization based on local defect-correction.

For elliptic boundary value problem, Xu and Zhou [21] combined two-grid finite element discretization scheme with the local defect-correction technique to propose a general and powerful parallel-computing technique. This technique has been used and developed by many scholars, for instance, successfully applied to Stokes equation (see [13, 14]), especially, Xu and Zhou [23], Dai and Zhou [8], Bi and Yang etc [4] developed this method and established local and parallel three-level finite element discretizations for symmetric elliptic singular

eigenvalue problems (including the electronic structure problems).

In this paper, we further apply local defect-correction technique proposed by Xu and Zhou to nonsymmetric elliptic singular eigenvalue problems, our work has the following features. (1) We extend local and parallel three-level finite element discretizations for symmetric eigenvalue problems established by Dai and Zhou [8] to nonsymmetric eigenvalue problems. (2) Based on [4], we establish new multilevel finite element discretization by local refinement, this scheme repeatedly makes defect correction on finer and finer local meshes to make up for abrupt changes of local mesh size caused by three level scheme. And theoretical analysis and numerical experiments show that our schemes are simple and easy to carry out, and can be used to solve singular nonsymmetric eigenvalue problems. Numerical experiments show that, compared with the adaptive homotopy approach in [5], our algorithm seems to be more efficient. (3) For the nonsymmetric problems, based on the work of [20, 21], we discuss the local error estimates of finite element approximations; its a new feature here that the estimates apply to the local domains containing corner points, see Lemmas 2.3-2.4 and Remark 2.2 in this paper.

In this paper, regarding the basic theory of finite elements, we refer to [1, 3, 7, 19].

## 2 preliminaries

Consider the nonsymmetric elliptic differential operator eigenvalue problem:

$$\begin{aligned} Lu &\equiv - \sum_{i,j=1}^d \partial_j(a_{ij}(x)\partial_i u) + \sum_{i=1}^d b_i(x)\partial_i u + c(x)u = \lambda m(x)u, \quad \text{in } \Omega, \quad (2.1) \\ u &= 0, \quad \text{on } \partial\Omega, \quad (2.2) \end{aligned}$$

where  $\Omega \subset R^d$ ,  $d \geq 2$ , be a polyhedral bounded domain with boundary  $\partial\Omega$ ,  $\partial_i u = \frac{\partial u}{\partial x_i}$ ,  $i = 1, 2, \dots, d$ .

Let

$$\begin{aligned} a(u, v) &= \int_{\Omega} \left( \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j \bar{v} + \sum_{i=1}^d b_i \partial_i u \bar{v} + c u \bar{v} \right) dx, \\ b(u, v) &= \int_{\Omega} m u \bar{v} dx. \end{aligned}$$

The variational form associated with (2.1)-(2.2) is given by: find  $\lambda \in \mathcal{C}$ ,  $u \in H_0^1(\Omega)$ ,  $\|u\|_0 = 1$ , satisfying

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega). \quad (2.3)$$

Assume that  $a_{i,j}, b_i \in W_{1,\infty}(\Omega)$ ,  $c \in L_{\infty}(\Omega)$  are given real or complex functions on  $\Omega$ ,  $m \in L_{\infty}(\Omega)$  is a given real function which is bounded below by a positive constant on  $\Omega$ .  $L$  is assumed to be uniformly strongly elliptic in  $\Omega$ , i.e.,

there is a positive constant  $a_0$  such that

$$\operatorname{Re} \sum_{i,j=1}^d a_{i,j} \xi_i \xi_j \geq a_0 \sum_{i=1}^d \xi_i^2, \quad \forall x \in \Omega, \quad \forall (\xi_1, \xi_2, \dots, \xi_d) \in R^d. \quad (2.4)$$

Let  $b = \max_{1 \leq i \leq d; x \in \Omega} |b_i(x)|$ . we assume without loss that  $\operatorname{Re} c \geq a_0/2 + b^2/(2a_0)$  since adding a constant  $\times m(x)$  to  $c(x)$  only shifts the eigenvalues. Under above assumptions, we have

$$\operatorname{Re} a(u, u) \geq \frac{1}{2} a_0 \|u\|_1^2, \quad \forall u \in H^1(\Omega); \quad (2.5)$$

and there are constants  $M_1$  and  $M_2$  such that

$$|a(u, v)| \leq M_1 \|u\|_1 \|v\|_1, \quad \forall u, v \in H^1(\Omega), \quad (2.6)$$

$$|b(u, v)| \leq M_2 \|u\|_0 \|v\|_0, \quad \forall u, v \in L_2(\Omega). \quad (2.7)$$

For  $D \subset \Omega_0 \subset \Omega$ , we use  $D \subset\subset \Omega_0$  to mean that  $\operatorname{dist}(\partial D \setminus \partial \Omega, \partial \Omega_0 \setminus \partial \Omega) > 0$ .

Assume that  $\pi_h(\Omega) = \{\tau\}$  is a mesh of  $\Omega$  with mesh-size function  $h(x)$  whose value is the diameter  $h_\tau$  of the element  $\tau$  containing  $x$ , and  $h(\Omega) = \max_{x \in \Omega} h(x)$  is the mesh diameter of  $\pi_h(\Omega)$ . We write  $h(\Omega)$  as  $h$  for simplicity.

Let  $V_h(\Omega) \subset C(\overline{\Omega})$ , defined on  $\pi_h(\Omega)$ , be a space of piecewise polynomials, and  $V_h^0(\Omega) = V_h(\Omega) \cap H_0^1(\Omega)$ . Given  $G \subset \Omega$ , we define  $\pi_h(G)$  and  $V_h(G)$  to be the restriction of  $\pi_h(\Omega)$  and  $V_h(\Omega)$  to  $G$ , respectively, and

$$V_h^0(G) = V_h(G) \cap H_0^1(G), \quad V_h^h(G) = \{v \in V_h^0(\Omega) : \operatorname{supp} v \subset\subset G\}.$$

For any  $G \subset \Omega$  mentioned in this paper, we assume that it aligns with  $\pi_h(\Omega)$  when necessary.

In this paper,  $C$  denotes a positive constant independent of  $h$ , which may not be the same constant in different places. For simplicity, we use the symbol  $a \lesssim b$  to mean that  $a \leq Cb$ .

We adopt the following assumptions in [21] for meshes and finite element space.

(A0) There exists  $\nu \geq 1$  such that  $h(\Omega)^\nu \lesssim h(x)$ ,  $\forall x \in \Omega$ .

(A1) There exists  $r \geq 1$  such that for  $w \in H_0^1(\Omega) \cap H^{1+t}(\Omega)$ ,

$$\inf_{v \in V_h^0(\Omega)} (\|h^{-1}(w - v)\|_0 + \|w - v\|_1) \lesssim h^t \|w\|_{1+t}, \quad 0 \leq t \leq r.$$

(A2) *Inverse Estimate.* For any  $v \in V_h(\Omega_0)$ ,  $\|v\|_{1, \Omega_0} \lesssim \|h^{-1}v\|_{0, \Omega_0}$ .

(A3) *Superapproximation.* For  $G \subset \Omega_0$ , let  $\omega \in C^\infty(\overline{\Omega})$  with  $\operatorname{supp} \omega \subset\subset G$ , then for any  $w \in V_h(G)$ ,  $w|_{\partial G \cap \partial \Omega} = 0$ , there exists  $v \in V_h^h(G)$  such that

$$\|h^{-1}(\omega w - v)\|_{1, G} \lesssim \|w\|_{1, G}.$$

Let  $\pi_h(\Omega)$  consist of shape-regular simplices and (A0) hold, and let  $V_h(\Omega) \subset C(\overline{\Omega})$  be a space of piecewise polynomials of degree  $\leq r$  defined on  $\pi_h(\Omega)$ , then from [21] we know that (A1)-(A3) are valid for this  $V_h(\Omega)$ .

The finite element approximation of (2.3) is given by: find  $\lambda_h \in C$ ,  $u_h \in V_h^0(\Omega)$ ,  $\|u_h\|_0 = 1$ , satisfying

$$a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in V_h^0(\Omega). \quad (2.8)$$

Thanks to [1], we know the adjoint problem of (2.1)-(2.2) is:

$$L^* u^* \equiv - \sum_{i,j=1}^d \partial_i (\bar{a}_{ij} \partial_j u^*) - \sum_{i=1}^d \partial_i (\bar{b}_i u^*) + \bar{c} u^* = \lambda^* m u^*, \quad \text{in } \Omega, \quad (2.9)$$

$$u^* = 0, \quad \text{on } \partial\Omega. \quad (2.10)$$

The corresponding variational form and discrete variational form of (2.9)-(2.10) are given by: find  $\lambda^* \in C$ ,  $u^* \in H_0^1(\Omega)$ ,  $\|u^*\|_0 = 1$ , satisfying

$$a(v, u^*) = \bar{\lambda}^* b(v, u^*), \quad \forall v \in H_0^1(\Omega); \quad (2.11)$$

find  $\lambda_h^* \in C$ ,  $u_h^* \in V_h$ ,  $\|u_h^*\|_0 = 1$ , satisfying

$$a(v, u_h^*) = \bar{\lambda}_h^* b(v, u_h^*), \quad \forall v \in V_h^0(\Omega). \quad (2.12)$$

Note that the primal and dual eigenvalues are connected via  $\lambda = \bar{\lambda}^*$  and  $\lambda_h = \bar{\lambda}_h^*$ .

Throughout this paper, we will assume that (2.5)-(2.7) hold. Thus from Lax-Milgram theorem we know the source problem associated with (2.3) and (2.11) admits an unique solution, respectively. The discrete source problem associated (2.8) and (2.12) admits an unique solution, respectively.

Define the solution operator  $T : L_2(\Omega) \rightarrow H_0^1(\Omega)$  and  $T_h : L_2(\Omega) \rightarrow V_h^0(\Omega)$  as follows:

$$a(Tg, v) = b(g, v), \quad \forall v \in H_0^1(\Omega), \quad (2.13)$$

$$a(T_h g, v) = b(g, v), \quad \forall v \in V_h^0(\Omega). \quad (2.14)$$

And (2.3) and (2.8) have the equivalent operator form (2.15) and (2.16), respectively.

$$Tu = \lambda^{-1} u, \quad (2.15)$$

$$T_h u_h = \lambda_h^{-1} u_h. \quad (2.16)$$

Define the solution operator  $T^* : L_2(\Omega) \rightarrow H_0^1(\Omega)$  and  $T_h^* : L_2(\Omega) \rightarrow V_h^0(\Omega)$  satisfying

$$a(v, T^* f) = b(v, f), \quad \forall v \in H_0^1(\Omega), \quad (2.17)$$

$$a(v, T_h^* f) = b(v, f), \quad \forall v \in V_h^0(\Omega). \quad (2.18)$$

And (2.11) and (2.12) have the equivalent operator forms (2.19) and (2.20), respectively.

$$T^* u^* = \lambda^{*-1} u^*, \quad (2.19)$$

$$T_h^* u_h^* = \lambda_h^{*-1} u_h^*. \quad (2.20)$$

It can be proved that  $T$  is completely continuous, and  $T^*$  is the adjoint operator of  $T$  in the sense of inner product  $b(\cdot, \cdot)$ . In fact,

$$\begin{aligned} b(Tu, v) &= a(Tu, T^*v) = b(u, T^*v), \quad \forall u, v \in L_2(\Omega), \\ b(T_h u, v) &= a(T_h u, T_h^* v) = b(u, T_h^* v), \quad \forall u, v \in L_2(\Omega). \end{aligned}$$

We need the following regularity assumption. For any  $f \in L_2(\Omega)$ ,  $Tf \in H_0^1(\Omega) \cap H^{1+\gamma_1}(\Omega)$  and  $T^*f \in H_0^1(\Omega) \cap H^{1+\gamma_2}(\Omega)$  satisfying

$$\|Tf\|_{1+\gamma_1} \leq C_\Omega \|f\|_0, \quad (2.21)$$

$$\|T^*f\|_{1+\gamma_2} \leq C_\Omega \|f\|_0. \quad (2.22)$$

According to [10] and the section 5.5 in [3], the above assumption is reasonable.

For some  $G \subset \Omega$ , we need the following local regularity assumption.

**R(G).** For any  $f \in L_2(G)$ , there exists a  $\phi \in H_0^1(G) \cap H^{1+\gamma_1}(G)$  satisfying

$$a(\phi, v) = b(f, v), \quad \forall v \in H_0^1(G),$$

and

$$\|\phi\|_{1+\gamma_1, G} \leq C_G \|f\|_{-1+\gamma_1, G}. \quad (2.23)$$

For any  $g \in L_2(G)$ , there exists a  $\varphi \in H_0^1(G) \cap H^{1+\gamma_2}(G)$  satisfying

$$a(v, \varphi) = b(v, g), \quad \forall v \in H_0^1(G),$$

and

$$\|\varphi\|_{1+\gamma_2, G} \leq C_G \|g\|_{-1+\gamma_2, G}. \quad (2.24)$$

Where  $C_\Omega$ ,  $C_G$  are two priori constants, and not necessarily the same at different places.

Define the Ritz projection  $P_h : H_0^1(\Omega) \rightarrow V_h^0(\Omega)$  and  $P_h^* : H_0^1(\Omega) \rightarrow V_h^0(\Omega)$  by

$$a(u - P_h u, v) = 0, \quad \text{and} \quad a(v, u - P_h^* u) = 0, \quad \forall v \in V_h^0(\Omega). \quad (2.25)$$

Then  $T_h = P_h T$ ,  $T_h^* = P_h^* T^*$  (see [1]).

Let  $M(\lambda)$  be the space spanned by all generalized eigenfunctions corresponding to  $\lambda$  of  $T$ ,  $M_h(\lambda)$  be the space spanned by all generalized eigenfunctions corresponding to all eigenvalues of  $T_h$  that converge to  $\lambda$ . In view of the adjoint problem (2.11) and (2.12), the definitions of  $M^*(\lambda^*)$  and  $M_h^*(\lambda^*)$  are analogous to  $M(\lambda)$  and  $M_h(\lambda)$ .

In this paper, we suppose that  $\lambda$  is an eigenvalue of (2.3) with the algebraic multiplicity  $q$  and the ascent  $\alpha = 1$ . Then  $\lambda^* = \bar{\lambda}$  be eigenvalue of (2.11),  $M(\lambda)$  and  $M^*(\lambda^*)$  are all eigenfunction space.

Let  $\lambda_h$  be the eigenvalue of (2.8) which converges to  $\lambda$ , let  $\lambda_h^* = \overline{\lambda_h}$ , and  $M^*(\lambda_h^*)$  be the generalized eigenfunction space corresponding to the eigenvalue  $\lambda_h^*$  of  $T_h^*$ .

**Remark 2.1.** Obviously, it's difficult to determine the ascent  $\alpha$  of the eigenvalue  $\lambda$  of (2.3) theoretically. But one could easily find that when the ascents of the eigenvalues of (2.8), which converge to the same eigenvalue  $\lambda$  of (2.3), are all equal to 1, one can conclude that the ascent  $\alpha = 1$  from the standard theory of spectral approximation. And the ascents of eigenvalues of (2.8) can be determined by computation.

We also need the lemma as follows (see [16, 25]):

**Lemma 2.1.** Let  $(\lambda, u)$  be an eigenpair of (2.3), and  $(\lambda^* = \bar{\lambda}, u^*)$  be the associated eigenpair of the adjoint problem (2.11). Then for all  $w, w^* \in H_0^1(\Omega)$ ,  $b(w, w^*) \neq 0$ ,

$$\frac{a(w, w^*)}{b(w, w^*)} - \lambda = \frac{a(w - u, w^* - u^*)}{b(w, w^*)} - \lambda \frac{b(w - u, w^* - u^*)}{b(w, w^*)}. \quad (2.26)$$

**Proof.** see [16, 25].  $\square$

The a priori error estimates of the finite element approximations (2.8) and (2.12) can refer to [1, 2].

**Lemma 2.2.** Assume that  $M(\lambda) \subset H^{r+s}(\Omega)$ ,  $M^*(\lambda^*) \subset H^{r+s_2}(\Omega)$  ( $0 < s, s_2 < 1$ ). Then

$$|\lambda_h - \lambda| \lesssim h^{r+s-1+r+s_2-1}, \quad (2.27)$$

let  $u_h \in M_h(\lambda)$  with  $\|u_h\|_0 = 1$ , then there is  $u \in M(\lambda)$  such that

$$\|u_h - u\|_1 \lesssim h^{r+s-1}, \quad (2.28)$$

$$\|u_h - u\|_0 \lesssim h^{r+s-1+\gamma_2}; \quad (2.29)$$

let  $u_h^* \in M_h^*(\lambda^*)$  with  $\|u_h^*\|_0 = 1$ , then there is  $u^* \in M^*(\lambda^*)$  such that

$$\|u_h^* - u^*\|_1 \lesssim h^{r+s_2-1}; \quad (2.30)$$

$$\|u_h^* - u^*\|_0 \lesssim h^{r+s_2-1+\gamma_1}. \quad (2.31)$$

**Proof.** see [1].  $\square$

[20, 21] etc. studied the local behavior of finite element. The following Lemma 2.3 is a simple generalization of Lemma 3.2 in [21]. We can easily prove this Lemma by the same argument as that of Lemma 3.2 in [21].

**Lemma 2.3.** Suppose that  $f \in H^{-1}(\Omega)$  and  $G \subset \subset \Omega_0 \subset \Omega$ . If  $w \in V_h(\Omega_0)$ ,  $w|_{\partial\Omega \cap \partial\Omega_0} = 0$ , satisfies

$$a(w, v) = f(v), \quad \forall v \in V_0^h(\Omega_0), \quad (2.32)$$

then

$$\|w\|_{1,G} \lesssim \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0}, \quad (2.33)$$

where

$$\|f\|_{-1,\Omega_0} = \sup_{\phi \in H_0^1(\Omega_0), \|\phi\|_{1,\Omega_0}=1} f(\phi).$$

**Proof.** Let  $p \geq 2\nu - 1$  be an integer, and let

$$D \subset\subset \Omega_p \subset\subset \Omega_{p-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Choose  $D_1 \subset \Omega$  satisfying  $D \subset\subset D_1 \subset\subset \Omega_p$  and  $\omega \in C^\infty(\bar{\Omega})$  such that  $\text{supp } \omega \subset\subset \Omega_p$  and  $\omega \equiv 1$  on  $\overline{D_1}$ . Then, from (A3), there exists  $v \in V_0^h(\Omega_p)$  such that

$$\|\omega^2 w - v\|_{1,\Omega_p} \lesssim h_{\Omega_0} \|w\|_{1,\Omega_p},$$

so we have

$$a(w, \omega^2 w - v) \lesssim h_{\Omega_0} \|w\|_{1,\Omega_p}^2 \quad (2.34)$$

and

$$\begin{aligned} |f(v)| &\lesssim \|f\|_{-1,\Omega_0} \|v\|_{1,\Omega_p} \\ &\lesssim \|f\|_{-1,\Omega_0} (h_{\Omega_0} \|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}). \end{aligned} \quad (2.35)$$

Since  $v \in V_0^h(\Omega_p) \subset V_0^h(\Omega_0)$ , (2.32) implies

$$a(w, \omega^2 w) = a(w, \omega^2 w - v) + f(v). \quad (2.36)$$

Let  $a_0(u, v) = \int_{\Omega} \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j \bar{v}$ . We can be derived from proof of Lemma 3.1 in [21] that if  $\Omega_0 \subset \Omega \subset R^d$  ( $d = 2, 3$ ),  $\omega \in C^\infty(\bar{\Omega})$ ,  $\text{supp } \omega \subset\subset \Omega_0$ , then

$$a_0(\omega w, \omega w) \lesssim a(w, \omega^2 w) + \|w\|_{0,\Omega_0}^2, \quad \forall w \in H_0^1(\Omega). \quad (2.37)$$

It follows from (2.34)-(2.37) that

$$\begin{aligned} \|\omega w\|_{1,\Omega}^2 &\lesssim a_0(\omega w, \omega w) \lesssim a(w, \omega^2 w) + \|w\|_{0,\Omega_0}^2 \\ &= a(w, \omega^2 w - v) + \|w\|_{0,\Omega_0}^2 + f(v) \\ &\lesssim h_{\Omega_0} \|w\|_{1,\Omega_p}^2 + \|w\|_{0,\Omega_0}^2 + \|f\|_{-1,\Omega_0} (h_{\Omega_0} \|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}), \end{aligned}$$

thus

$$\|w\|_{1,D} \lesssim h_{\Omega_0}^{1/2} \|w\|_{1,\Omega_p} + \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0}. \quad (2.38)$$

Similarly, we can get

$$\|w\|_{1,\Omega_j} \lesssim h_{\Omega_0}^{1/2} \|w\|_{1,\Omega_{j-1}} + \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0}, \quad j = 1, 2, \dots, p. \quad (2.39)$$

By using (2.38) and (2.39), we get from (A0) and (A2) that

$$\begin{aligned} \|w\|_{1,D} &\lesssim h_{\Omega_0}^{(p+1)/2} \|w\|_{1,\Omega_0} + \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0} \\ &\lesssim h_{\Omega_0}^{(p+1)/2} \|h^{-1} w\|_{0,\Omega_0} + \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0} \\ &\lesssim \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 2.4.** Suppose that  $G \subset\subset \Omega_0 \subset \Omega$ . Then the following estimates are valid:

$$h^{\gamma_2} \|u - P_h u\|_{1,\Omega} + \|u - P_h u\|_{0,\Omega} \lesssim h^{\gamma_2} \inf_{v \in V_h^0(\Omega)} \|u - v\|_{1,\Omega}, \quad (2.40)$$

$$\|u - P_h u\|_{1,G} \lesssim \inf_{v \in V_h^0(\Omega)} \|u - v\|_{1,\Omega_0} + h^{\gamma_2} \|u - P_h u\|_{1,\Omega}. \quad (2.41)$$

**Proof.** For proof of (2.40) cf. [3, 7], for proof of (2.41) cf. Theorem 3.4 in [21].  $\square$

**Remark 2.2.** In [21], the condition *Superapproximation* is given as follows.

**A.3. Superapproximation.** For  $G \subset \Omega_0$ , let  $\omega \in C_0^\infty(\Omega)$  with  $\text{supp } \omega \subset\subset G$ . Then for any  $w \in V_h(G)$ , there exists  $v \in V_0^h(G)$  such that

$$\|h^{-1}(\omega w - v)\|_{1,G} \lesssim \|w\|_{1,G}.$$

In the proof of Lemma 3.2 in [21] the authors choose  $D_1 \subset \Omega$  satisfying  $D \subset\subset D_1 \subset\subset \Omega_p$  and  $\omega \in C_0^\infty(\Omega)$  such that  $\omega \equiv 1$  on  $\overline{D_1}$  and  $\text{supp } \omega \subset\subset \Omega_p$ . This paper just makes a minor modification, so that the theory of the local error estimates built in [21] applies to the local domains containing the corner points, see Lemma 2.3 and Lemma 2.4.

### 3 Multilevel discretizations based on local defect-correction

Consider the eigenvalue problem (2.3) which has an isolated singular point  $z \in \overline{\Omega}$  (e.g., see Figure 3.1).

Let  $D \subset\subset \Omega$  be a given subdomain containing the singular point  $z$ , and we introduce domains

$$\Omega \supset \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_l \supset\supset D.$$

Let  $\pi_H(\Omega)$  be a shape-regular grid, which is made up of simplices, with size  $H \in (0, 1)$ ,  $\pi_w(\Omega)$  be a refined mesoscopic shape-regular grid (from  $\pi_H(\Omega)$ ) and  $\pi_{h_i}(\Omega_i)$  be a locally refined grid (from  $\pi_{h_{i-1}}(\Omega_{i-1})$ ) that satisfy  $h_{-1} = H$ ,  $h_0 = w$ ,  $h_i \ll h_{i-1}$  ( $i = 0, 1, \dots, l$ ). (Figure 3.1 shows  $\pi_H(\Omega)$ ,  $\pi_w(\Omega)$  and  $\pi_{h_1}(\Omega_1)$ ). Let  $V_H^0(\Omega)$ ,  $V_w^0(\Omega)$ , and  $\{V_{h_i}^0(\Omega_i)\}_1^l$  be finite element spaces of degree less than or equal to  $r$  defined on  $\pi_H(\Omega)$ ,  $\pi_w(\Omega)$  and  $\{\pi_{h_i}(\Omega_i)\}_1^l$ , respectively.

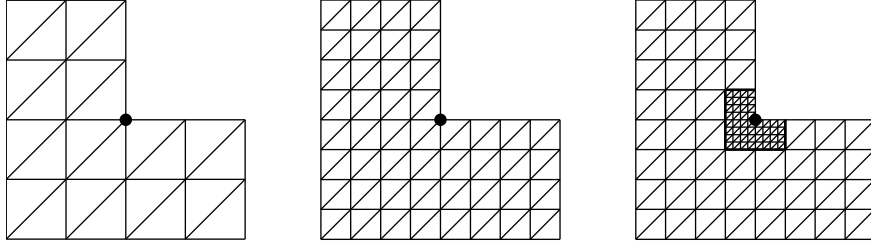


Figure 3.1



Based on Algorithm  $B_0$  in [8] we establish the following three-level discretization scheme.

**Scheme 3.1** (Three-level discretizations based on local defect-correction.).

**Step 1.** Solve (2.3) on a globally coarse grid  $\pi_H(\Omega)$ : find  $\lambda_H \in \mathcal{C}$ ,  $u_H \in V_H^0(\Omega)$  such that  $\|u_H\|_0 = 1$  and

$$a(u_H, v) = \lambda_H b(u_H, v), \quad \forall v \in V_H^0(\Omega).$$

Let  $\lambda_H^* = \overline{\lambda}_H$ , and find  $u_H^* \in M^*(\lambda_H^*)$  with  $\|u_H^*\|_0 = 1$  such that  $|b(u_H, u_H^*)|$  has a positive lower bound uniformly with respect to  $H$  (see section 5.1).

**Step 2.** Solve two linear boundary value problems on a globally mesoscopic grid  $\pi_w(\Omega)$ : find  $u^w \in V_w^0(\Omega)$  such that

$$a(u^w, v) = \lambda_H b(u_H, v), \quad \forall v \in V_w^0(\Omega);$$

find  $u^{w*} \in V_w^0(\Omega)$  such that

$$a(v, u^{w*}) = \lambda_H b(v, u_H^*), \quad \forall v \in V_w^0(\Omega).$$

Then compute the Rayleigh quotient  $\lambda^w = \frac{a(u^w, u^{w*})}{b(u^w, u^{w*})}$ .

**Step 3.** Solve two linear boundary value problems on a locally fine grid  $\pi_{h_1}(\Omega_1)$ : find  $e^{h_1} \in V_{h_1}^0(\Omega_1)$  such that

$$a(e^{h_1}, v) = \lambda^w b(u^w, v) - a(u^w, v), \quad \forall v \in V_{h_1}^0(\Omega_1); \quad (3.1)$$

find  $e^{h_1*} \in V_{h_1}^0(\Omega_1)$  such that

$$a(v, e^{h_1*}) = \lambda^w b(v, u^{w*}) - a(v, u^{w*}), \quad \forall v \in V_{h_1}^0(\Omega_1). \quad (3.2)$$

**Step 4.** Set

$$u^{w, h_1} = \begin{cases} u^w + e^{h_1} & \text{on } \overline{\Omega}_1, \\ u^w & \text{in } \Omega \setminus \overline{\Omega}_1 \end{cases} \quad (3.3)$$

$$u^{w, h_1*} = \begin{cases} u^{w*} + e^{h_1*} & \text{on } \overline{\Omega}_1, \\ u^{w*} & \text{in } \Omega \setminus \overline{\Omega}_1 \end{cases} \quad (3.4)$$

And compute the Rayleigh quotient

$$\lambda^{w, h_1} = \frac{a(u^{w, h_1}, u^{w, h_1*})}{b(u^{w, h_1}, u^{w, h_1*})}, \quad \lambda^{w, h_1*} = \overline{\lambda^{w, h_1}}. \quad (3.5)$$

We use  $(\lambda^{w, h_1}, u^{w, h_1})$  and  $(\lambda^{w, h_1*}, u^{w, h_1*})$  obtained by Scheme 3.1 as the approximate eigenpair of (2.3) and (2.11), respectively.

It is obvious that  $(\lambda^w, u^w)$  and  $(\overline{\lambda^w}, u^{w*})$  in Scheme 3.1 can be viewed as approximate eigenpairs obtained by the two-grid discretization scheme in [16, 25] from  $\pi_H(\Omega)$  and  $\pi_w(\Omega)$ .

Using Scheme 3.1, abrupt changes of mesh size can appear near  $\partial\Omega_1$ . Influenced by the technique on the transition layer proposed by [4], we repeatedly use the local defect-correction technique to establish the following multilevel discretization scheme.

**Scheme 3.2** (multilevel discretizations based on local defect-correction.).

**Step 1.** The same as that of Step 1 of Scheme 3.1.

**Step 2.** The same as that of Step 2 of Scheme 3.1.

**Step 3.**  $u^{w,h_0} \Leftarrow u^w$ ,  $\lambda^{w,h_0} \Leftarrow \lambda^w$ ,  $u^{w,h_0*} \Leftarrow u^{w*}$ ,  $\lambda^{w,h_0*} \Leftarrow \lambda^{w*}$ .

**Step 4.** For  $i = 1, 2, \dots, l$ , execute Step 5 and Step 6.

**Step 5.** Solve linear boundary value problems on locally fine grid  $\pi_{h_i}(\Omega_i)$ : find  $e^{h_i} \in V_{h_i}^0(\Omega_i)$  such that

$$a(e^{h_i}, v) = \lambda^{w,h_{i-1}} b(u^{w,h_{i-1}}, v) - a(u^{w,h_{i-1}}, v), \quad \forall v \in V_{h_i}^0(\Omega_i); \quad (3.6)$$

find  $e^{h_i*} \in V_{h_i}^0(\Omega_i)$  such that

$$a(v, e^{h_i*}) = \lambda^{w,h_{i-1}*} b(v, u^{w,h_{i-1}*}) - a(v, u^{w,h_{i-1}*}), \quad \forall v \in V_{h_i}^0(\Omega_i). \quad (3.7)$$

**Step 6.** Set

$$u^{w,h_i} = \begin{cases} u^{w,h_{i-1}} + e^{h_i} & \text{on } \overline{\Omega}_i, \\ u^{w,h_{i-1}} & \text{in } \Omega \setminus \overline{\Omega}_i \end{cases} \quad (3.8)$$

$$u^{w,h_i*} = \begin{cases} u^{w,h_{i-1}*} + e^{h_i*} & \text{on } \overline{\Omega}_i, \\ u^{w,h_{i-1}*} & \text{in } \Omega \setminus \overline{\Omega}_i \end{cases} \quad (3.9)$$

And compute

$$\lambda^{w,h_i} = \frac{a(u^{w,h_i}, u^{w,h_i*})}{b(u^{w,h_i}, u^{w,h_i*})}, \quad \lambda^{w,h_i*} = \overline{\lambda^{w,h_i}}. \quad (3.10)$$

We use  $(\lambda^{w,h_i}, u^{w,h_i})$  and  $(\lambda^{w,h_i*}, u^{w,h_i*})$  obtained by Scheme 3.2 as the approximate eigenpair of (2.3) and (2.11), respectively.

## 4 Theoretical Analysis

Next we shall discuss the error estimates of Scheme 3.1 and Scheme 3.2.

In our analysis, we introduce an auxiliary grid  $\pi_{h_i}(\Omega)$  which is defined globally, and denote the piecewise polynomials space of degree  $\leq r$  by  $V_{h_i}^0(\Omega)$  ( $i = 1, 2, \dots, l$ ). We also assume that  $\pi_{h_i}(\Omega_i)$  and  $V_{h_i}^0(\Omega_i)$  are the restrictions of  $\pi_{h_i}(\Omega)$  and  $V_{h_i}^0(\Omega)$  to  $\Omega_i$ , respectively, and

$$V_H^0(\Omega) \subset V_w^0(\Omega) \subset V_{h_1}^0(\Omega) \subset V_{h_2}^0(\Omega) \subset \dots \subset V_{h_l}^0(\Omega).$$

For  $D$  and  $\Omega_i$  stated at the beginning of section 3, let  $G_i \subset \Omega$  and  $F \subset \Omega$  satisfy  $D \subset \subset F \subset \subset G_i \subset \subset \Omega_i$  ( $i = 1, 2, \dots, l$ ).

**Theorem 4.1.** Assume that  $M(\lambda) \subset H_0^1(\Omega) \cap H^{r+s}(\Omega) \cap H^{r+1}(\Omega \setminus \overline{D})$  and  $(1 < r + s, 0 \leq s < 1)$ ,  $M^*(\lambda^*) \subset H_0^1(\Omega) \cap H^{r+s_2}(\Omega) \cap H^{r+1}(\Omega \setminus \overline{D})$  and  $(1 < r + s_2, 0 \leq s_2 < 1)$ , and  $H$  is properly small. Then there exists  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$  such that

$$\|u^w - u\|_1 \lesssim H^{r+s-1+\gamma_2} + w^{r+s-1}, \quad (4.1)$$

$$\|u^w - u\|_0 \lesssim H^{r+s-1+\gamma_2}, \quad (4.2)$$

$$\|u^w - u\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma_2} + w^r, \quad (4.3)$$

$$\|u^{w*} - u^*\|_1 \lesssim H^{r+s_2-1+\gamma_1} + w^{r+s_2-1}, \quad (4.4)$$

$$\|u^{w*} - u^*\|_0 \lesssim H^{r+s_2-1+\gamma_1}, \quad (4.5)$$

$$\|u^{w*} - u^*\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s_2-1+\gamma_1} + w^r, \quad (4.6)$$

$$|\lambda^w - \lambda| \lesssim H^{2r+s+s_2-2+\gamma_1+\gamma_2} + w^{2r+s+s_2-2}. \quad (4.7)$$

**Proof.** Let  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$  such that  $u - u_H$  and  $u^* - u_H^*$  both satisfy Lemma 2.2. From (2.13), (2.14), Step 2 of Scheme 3.1, (2.15), Lemma 2.2 and Lemma 2.4, we derive that

$$\begin{aligned} \|u^w - u\|_1 &= \|\lambda_H T_w u_H - \lambda T u\|_1 \\ &\leq \|\lambda_H T_w u_H - \lambda T_w u\|_1 + \|\lambda T_w u - \lambda T u\|_1 \\ &\lesssim \|\lambda_H u_H - \lambda u\|_0 + \lambda \|P_w T u - T u\|_1 \\ &\lesssim H^{r+s-1+\gamma_2} + w^{r+s-1}, \end{aligned}$$

then (4.1) follows. By Lemma 2.2 and Lemma 2.4,

$$\begin{aligned} \|u^w - u\|_{1, \Omega \setminus \overline{F}} &\lesssim \|\lambda_H u_H - \lambda u\|_0 + \lambda \|P_w T u - T u\|_{1, \Omega \setminus \overline{F}} \\ &\lesssim H^{r+s-1+\gamma_2} + w^r, \end{aligned}$$

then (4.3) follows. By calculation,

$$\begin{aligned} \|u^w - u\|_0 &= \|\lambda_H T_w u_H - \lambda T u\|_0 \\ &\leq \|\lambda_H T_w u_H - \lambda T_w u\|_0 + \|\lambda T_w u - \lambda T u\|_0 \\ &\lesssim \|\lambda_H u_H - \lambda u\|_0 + \lambda \|P_w T u - T u\|_0 \\ &\lesssim H^{r+s-1+\gamma_2} + w^{r+s-1+\gamma_2} \\ &\lesssim H^{r+s-1+\gamma_2}, \end{aligned}$$

then (4.2) follows.

Similarly we can prove (4.4), (4.5) and (4.6). From (2.26), we have

$$\lambda^w - \lambda = \frac{a(u^w - u, u^{w*} - u^*)}{b(u^w, u^{w*})} - \lambda \frac{b(u^w - u, u^{w*} - u^*)}{b(u^w, u^{w*})}. \quad (4.8)$$

Note that  $u_H$  and  $u^w$  just approximate the same eigenfunction  $u$ ,  $u_H^*$  and  $u^{w*}$  approximate the same adjoint eigenfunction  $u^*$ ,  $|b(u_H, u_H^*)|$  has a positive lower bound uniformly with respect to  $H$ , therefore  $b(u^w, u^{w*})$  has a positive lower bound uniformly. Combining (4.1), (4.2), (4.4), (4.5) and (4.8) yields (4.7).  $\square$

The following Theorem 4.2 is a critical result in this paper, which develops the results of Theorem 3.3 in [8].

**Theorem 4.2.** Assume that  $R(\Omega_i)$  holds ( $i = 1, 2, \dots, l$ ),  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$ . Then

$$\begin{aligned} \|u^{w, h_l} - P_{h_l} u\|_{1, \Omega} &\lesssim \|u - P_{h_l} u\|_{0, \Omega_l} + h_{l-1}^{\gamma_2} \|P_{h_l} u - u^{w, h_{l-1}}\|_{1, \Omega_l} \\ &\quad + \|\lambda u - \lambda^{w, h_{l-2}} u^{w, h_{l-2}}\|_{0, \Omega_l} + \|\lambda^{w, h_{l-1}} u^{w, h_{l-1}} - \lambda u\|_0 \\ &\quad + \|u^{w, h_{l-1}} - P_{h_l} u\|_{1, \Omega \setminus \overline{G_l}} + \|u^{w, h_{l-1}} - u\|_{1, \Omega_l \setminus \overline{F}}, \quad l \geq 1; \end{aligned} \quad (4.9)$$

$$\begin{aligned} \|u^{w, h_l^*} - P_{h_l}^* u^*\|_{1, \Omega} &\lesssim \|u^* - P_{h_l}^* u^*\|_{0, \Omega_l} + h_{l-1}^{\gamma_1} \|P_{h_l}^* u^* - u^{w, h_{l-1}^*}\|_{1, \Omega_l} \\ &\quad + \|\lambda u^* - \lambda^{w, h_{l-2}^*} u^{w, h_{l-2}^*}\|_{0, \Omega_l} + \|\lambda^{w, h_{l-1}^*} u^{w, h_{l-1}^*} - \lambda u^*\|_0 \\ &\quad + \|u^{w, h_{l-1}^*} - P_{h_l}^* u^*\|_{1, \Omega \setminus \overline{G_l}} + \|u^{w, h_{l-1}^*} - u^*\|_{1, \Omega_l \setminus \overline{F}}, \quad l \geq 1. \end{aligned} \quad (4.10)$$

**Proof.** Due to the inequality

$$\begin{aligned} \|u^{w, h_l} - P_{h_l} u\|_{1, \Omega} &\lesssim \|u^{w, h_l} - P_{h_l} u\|_{1, D} + \|u^{w, h_l} - P_{h_l} u\|_{1, G_l \setminus \overline{D}} \\ &\quad + \|u^{w, h_l} - P_{h_l} u\|_{1, \Omega \setminus \overline{G_l}}, \end{aligned} \quad (4.11)$$

we shall estimate  $\|u^{w,h_l} - P_{h_l}u\|_{1,D}$ ,  $\|u^{w,h_l} - P_{h_l}u\|_{1,G_l \setminus \overline{D}}$ , and  $\|u^{w,h_l} - P_{h_l}u\|_{1,\Omega \setminus \overline{G_l}}$ , respectively.

First, we proceed to estimate  $\|u^{w,h_l} - P_{h_l}u\|_{1,D}$ . From (3.8), (3.6) and (2.25) we derive

$$\begin{aligned} a(u^{w,h_l} - P_{h_l}u, v) &= a(u^{w,h_l}, v) - a(P_{h_l}u, v) = a(u^{w,h_{l-1}} + e^{h_l}, v) - a(u, v) \\ &= \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}}, v) - \lambda b(u, v), \quad \forall v \in V_{h_l}^0(\Omega_l). \end{aligned} \quad (4.12)$$

It is obvious that

$$\begin{aligned} &\lambda^{w,h_{l-1}}b(u^{w,h_{l-1}}, v) - \lambda b(u, v) \\ &= (\lambda^{w,h_{l-1}} - \lambda)b(u, v) + \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}} - u, v), \quad \forall v \in H_0^1(\Omega) \end{aligned} \quad (4.13)$$

which together with (4.12) yields

$$a(u^{w,h_l} - P_{h_l}u, v) = (\lambda^{w,h_{l-1}} - \lambda)b(u, v) + \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}} - u, v), \quad \forall v \in V_{h_l}^0(\Omega_l).$$

Since  $V_{h_l}^{h_l}(\Omega_l) \subset V_{h_l}^0(\Omega_l)$ , thus, from the above formula and Lemma 2.3 we deduce that

$$\|u^{w,h_l} - P_{h_l}u\|_{1,D} \lesssim \|u^{w,h_l} - P_{h_l}u\|_{0,\Omega_l} + |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_l} - u\|_{0,\Omega_l} \quad (4.14)$$

By calculation, we have

$$\begin{aligned} \|u^{w,h_l} - P_{h_l}u\|_{0,\Omega_l} &\leq \|u^{w,h_{l-1}} - P_{h_l}u\|_{0,\Omega_l} + \|e^{h_l}\|_{0,\Omega_l} \\ &\leq \|u - P_{h_l}u\|_{0,\Omega_l} + \|u - u^{w,h_{l-1}}\|_{0,\Omega_l} + \|e^{h_l}\|_{0,\Omega_l}, \end{aligned}$$

substituting the above relation into (4.14) we obtain

$$\begin{aligned} &\|u^{w,h_l} - P_{h_l}u\|_{1,D} \\ &\lesssim |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_{l-1}} - u\|_{0,\Omega_l} + \|u - P_{h_l}u\|_{0,\Omega_l} + \|e^{h_l}\|_{0,\Omega_l} \end{aligned} \quad (4.15)$$

To estimate  $\|e^{h_l}\|_{0,\Omega_l}$ , we use the Aubin-Nitsche duality argument. For any given  $f \in L_2(\Omega_l)$ , consider the boundary value problem: find  $\varphi \in H_0^1(\Omega_l)$  such that

$$a(v, \varphi) = b(v, f) \quad \forall v \in H_0^1(\Omega_l). \quad (4.16)$$

Let  $\varphi$  be the generalized solution of (4.16),  $\varphi_{h_l}$  and  $\varphi_{h_{l-1}}$  be finite element solutions of (4.16) in  $V_{h_l}^0(\Omega_l)$  and  $V_{h_{l-1}}^0(\Omega_l)$ , respectively. Then,

$$\|\varphi - \varphi_{h_l}\|_{1,\Omega_l} \lesssim h_l^{\gamma_2} \|f\|_{0,\Omega_l}, \quad \|\varphi - \varphi_{h_{l-1}}\|_{1,\Omega_l} \lesssim h_{l-1}^{\gamma_2} \|f\|_{0,\Omega_l}. \quad (4.17)$$

From (3.6) and (3.8) we get

$$a(u^{w,h_l}, \varphi_{h_l}) = \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}}, \varphi_{h_l}),$$

thus by the definitions of  $\varphi$ ,  $\varphi_{h_l}$  and  $e^{h_l}$ , we deduce that

$$\begin{aligned} b(e^{h_l}, f) &= a(e^{h_l}, \varphi) = a(e^{h_l}, \varphi_{h_l}) = a(u^{w,h_l} - u^{w,h_{l-1}}, \varphi_{h_l}) \\ &= a(P_{h_l}u - u^{w,h_{l-1}}, \varphi_{h_l}) + a(u^{w,h_l}, \varphi_{h_l}) - a(P_{h_l}u, \varphi_{h_l}) \\ &= a(P_{h_l}u - u^{w,h_{l-1}}, \varphi_{h_l}) + \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}}, \varphi_{h_l}) - \lambda b(u, \varphi_{h_l}) \\ &= a(P_{h_l}u - u^{w,h_{l-1}}, \varphi_{h_l} - \varphi) + a(P_{h_l}u - u^{w,h_{l-1}}, \varphi - \varphi_{h_{l-1}}) \\ &\quad + a(P_{h_l}u - u^{w,h_{l-1}}, \varphi_{h_{l-1}}) + \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}}, \varphi_{h_l}) - \lambda b(u, \varphi_{h_l}) \\ &\lesssim h_{l-1}^{\gamma_2} \|P_{h_l}u - u^{w,h_{l-1}}\|_{1,\Omega_l} \|f\|_{0,\Omega_l} + a(P_{h_l}u - u^{w,h_{l-1}}, \varphi_{h_{l-1}}) \\ &\quad + \lambda^{w,h_{l-1}}b(u^{w,h_{l-1}}, \varphi_{h_l}) - \lambda b(u, \varphi_{h_l}). \end{aligned} \quad (4.18)$$

Step 2 of Scheme 3.2 shows that

$$a(u^{w,h_0}, \varphi_{h_0}) = \lambda^{w,h-1} b(u^{w,h-1}, \varphi_{h_0}),$$

namely, for  $l = 1$ ,

$$a(u^{w,h_{l-1}}, \varphi_{h_{l-1}}) = \lambda^{w,h_{l-2}} b(u^{w,h_{l-2}}, \varphi_{h_{l-1}}),$$

for  $l > 1$ , the above formula follows from (3.6) and (3.8). Therefore,

$$\begin{aligned} a(P_{h_l} u - u^{w,h_{l-1}}, \varphi_{h_{l-1}}) &= a(u - u^{w,h_{l-1}}, \varphi_{h_{l-1}}) \\ &= \lambda b(u, \varphi_{h_{l-1}}) - a(u^{w,h_{l-1}}, \varphi_{h_{l-1}}) \\ &= \lambda b(u, \varphi_{h_{l-1}}) - \lambda^{w,h_{l-2}} b(u^{w,h_{l-2}}, \varphi_{h_{l-1}}) \\ &\lesssim \|\lambda u - \lambda^{w,h_{l-2}} u^{w,h_{l-2}}\|_{0,\Omega_l} \|f\|_{0,\Omega_l}. \end{aligned}$$

It is clear that

$$|\lambda^{w,h_{l-1}} b(u^{w,h_{l-1}}, \varphi_{h_l}) - \lambda b(u, \varphi_{h_l})| \lesssim \|\lambda^{w,h_{l-1}} u^{w,h_{l-1}} - \lambda u\|_{0,\Omega_l} \|f\|_{0,\Omega_l}.$$

Substituting the above two formulae into (4.18), we derive

$$\begin{aligned} |b(e^{h_l}, f)| &\lesssim (h_{l-1}^{\gamma_2} \|P_{h_l} u - u^{w,h_{l-1}}\|_{1,\Omega_l} + \|\lambda u - \lambda^{w,h_{l-2}} u^{w,h_{l-2}}\|_{0,\Omega_l} \\ &\quad + \|\lambda^{w,h_{l-1}} u^{w,h_{l-1}} - \lambda u\|_0) \|f\|_{0,\Omega_l}. \end{aligned}$$

Thus, we get

$$\begin{aligned} \|e^{h_l}\|_{0,\Omega_l} &\lesssim h_{l-1}^{\gamma_2} \|P_{h_l} u - u^{w,h_{l-1}}\|_{1,\Omega_l} + \|\lambda u - \lambda^{w,h_{l-2}} u^{w,h_{l-2}}\|_{0,\Omega_l} \\ &\quad + \|\lambda^{w,h_{l-1}} u^{w,h_{l-1}} - \lambda u\|_0. \end{aligned} \quad (4.19)$$

Substituting (4.19) into (4.15), we obtain

$$\begin{aligned} \|u^{w,h_l} - P_{h_l} u\|_{1,D} &\lesssim \|u - P_{h_l} u\|_{0,\Omega_l} + h_{l-1}^{\gamma_2} \|P_{h_l} u - u^{w,h_{l-1}}\|_{1,\Omega_l} \\ &\quad + \|\lambda u - \lambda^{w,h_{l-2}} u^{w,h_{l-2}}\|_{0,\Omega_l} + |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_{l-1}} - u\|_0. \end{aligned} \quad (4.20)$$

Similarly, since  $(G_l \setminus \overline{D}) \subset \subset \Omega_l$ , we deduce

$$\begin{aligned} \|u^{w,h_l} - P_{h_l} u\|_{1,D} &\lesssim \|u - P_{h_l} u\|_{0,\Omega_l} + h_{l-1}^{\gamma_2} \|P_{h_l} u - u^{w,h_{l-1}}\|_{1,\Omega_l} \\ &\quad + \|\lambda u - \lambda^{w,h_{l-2}} u^{w,h_{l-2}}\|_{0,\Omega_l} + |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_{l-1}} - u\|_0. \end{aligned} \quad (4.21)$$

The remainder is to analyze  $\|u^{w,h_l} - P_{h_l} u\|_{1,\Omega \setminus \overline{G}}$ . From (3.8), we see that

$$\|u^{w,h_l} - P_{h_l} u\|_{1,\Omega \setminus \overline{\Omega_l}} = \|u^{w,h_{l-1}} - P_{h_l} u\|_{1,\Omega \setminus \overline{\Omega_l}},$$

which leads to

$$\begin{aligned} &\|u^{w,h_l} - P_{h_l} u\|_{1,\Omega \setminus \overline{G_l}} \\ &\leq \|u^{w,h_l} - P_{h_l} u\|_{1,\Omega \setminus \overline{\Omega_l}} + \|u^{w,h_{l-1}} - P_{h_l} u\|_{1,\Omega_l \setminus \overline{G_l}} + \|e^{h_l}\|_{1,\Omega_l \setminus \overline{G_l}} \\ &\lesssim \|u^{w,h_{l-1}} - P_{h_l} u\|_{1,\Omega \setminus \overline{G_l}} + \|e^{h_l}\|_{1,\Omega_l \setminus \overline{G_l}}. \end{aligned} \quad (4.22)$$

It follows from (3.6), (2.3) and (4.13) that

$$\begin{aligned} a(e^{h_l}, v) &= \lambda^{w,h_{l-1}} b(u^{w,h_{l-1}}, v) - a(u^{w,h_{l-1}}, v) - \lambda b(u, v) + a(u, v) \\ &= (\lambda^{w,h_{l-1}} - \lambda) b(u, v) + \lambda^{w,h_{l-1}} b(u^{w,h_{l-1}} - u, v) \\ &\quad - a(u^{w,h_{l-1}} - u, v), \quad \forall v \in V_h^0(\Omega_l), \end{aligned}$$

then, by Lemma 2.3, we have

$$\|e^{h_l}\|_{1,\Omega_l\setminus\overline{G_l}} \lesssim \|e^{h_l}\|_{0,\Omega_l\setminus\overline{F}} + |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_{l-1}} - u\|_{1,\Omega_l\setminus\overline{F}}, \quad (4.23)$$

where  $F \subset \Omega$  satisfies  $D \subset\subset F \subset\subset G_l$ . Substituting (4.23) into (4.22) we get

$$\begin{aligned} & \|u^{w,h_l} - P_{h_l}u\|_{1,\Omega\setminus\overline{G_l}} \\ & \lesssim \|u^{w,h_{l-1}} - P_{h_l}u\|_{1,\Omega\setminus\overline{G_l}} + \|e^{h_l}\|_{0,\Omega_l\setminus\overline{F}} + |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_{l-1}} - u\|_{1,\Omega_l\setminus\overline{F}}. \end{aligned}$$

It follows from substituting (4.19) into the above inequality that

$$\begin{aligned} \|u^{w,h_l} - P_{h_l}u\|_{1,\Omega\setminus\overline{G_l}} & \lesssim \|u^{w,h_{l-1}} - P_{h_l}u\|_{1,\Omega\setminus\overline{G_l}} + h_{l-1}^{\gamma_2} \|P_{h_l}u - u^{w,h_{l-1}}\|_{1,\Omega_l} \\ & + \|\lambda u - \lambda^{w,h_{l-2}}u^{w,h_{l-2}}\|_{0,\Omega_l} + \|\lambda^{w,h_{l-1}}u^{w,h_{l-1}} - \lambda u\|_0 \\ & + |\lambda^{w,h_{l-1}} - \lambda| + \|u^{w,h_{l-1}} - u\|_{1,\Omega_l\setminus\overline{F}}. \end{aligned} \quad (4.24)$$

Combining (4.24), (4.20), (4.21) and (4.11), finally, we obtain (4.9).

We can prove (4.10) by using the similar argument.  $\square$

**Theorem 4.3.** Assume that the conditions of Theorem 4.1 hold. Then there exists  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$  such that

$$\|u^{w,h_1} - u\|_{1,\Omega} \lesssim h_1^{r+s-1} + w^r + H^{r+s-1+\gamma_2}, \quad (4.25)$$

$$\|u^{w,h_1} - u\|_{0,\Omega} \lesssim w^r + H^{r+s-1+\gamma_2}, \quad (4.26)$$

$$\|u^{w,h_1} - u\|_{1,\Omega\setminus\overline{F}} \lesssim w^r + H^{r+s-1+\gamma_2}, \quad (4.27)$$

$$\|u^{w,h_1^*} - u^*\|_{1,\Omega} \lesssim h_1^{r+s_2-1} + w^r + H^{r+s_2-1+\gamma_1}, \quad (4.28)$$

$$\|u^{w,h_1^*} - u^*\|_{0,\Omega} \lesssim w^r + H^{r+s_2-1+\gamma_1}, \quad (4.29)$$

$$\|u^{w,h_1^*} - u^*\|_{1,\Omega\setminus\overline{F}} \lesssim w^r + H^{r+s_2-1+\gamma_1}, \quad (4.30)$$

$$|\lambda^{w,h_1} - \lambda| \lesssim h_1^{2r+s+s_2-2} + w^{2r} + H^{2r+s+s_2-2+\gamma_1+\gamma_2}. \quad (4.31)$$

**Proof.** Let  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$  such that  $u - u_H$  and  $u^* - u_H^*$  both satisfy Lemma 2.2. In Theorem 4.2, choose  $l = 1$ ,  $h_{-1} = H$ ,  $h_0 = w$ ,  $u^{w,h_0} = u^w$ ,  $\lambda^{w,h_0} = \lambda^w$ ,  $u^{w,h_{-1}} = u_H$ ,  $\lambda^{w,h_{-1}} = \lambda_H$ , then we get

$$\begin{aligned} \|u^{w,h_1} - P_{h_1}u\|_{1,\Omega} & \lesssim \|u - P_{h_1}u\|_{0,\Omega_1} + w^{\gamma_2} \|P_{h_1}u - u^w\|_{1,\Omega_1} \\ & + \|\lambda u - \lambda_H u_H\|_{0,\Omega_1} + \|\lambda^w u^w - \lambda u\|_0 \\ & + \|u^w - P_{h_1}u\|_{1,\Omega\setminus\overline{G_1}} + \|u^w - u\|_{1,\Omega_1\setminus\overline{F}}. \end{aligned} \quad (4.32)$$

Using Lemma 2.4, Theorem 4.1, Lemma 2.2 to estimate the terms at the right hand side of the above formula gives

$$\begin{aligned} \|u^{w,h_1} - P_{h_1}u\|_{1,\Omega} & \lesssim h_1^{r+s-1+\gamma_2} + w^{\gamma_2} w^{r+s-1} + H^{r+s-1+\gamma_2} + w^{r+s-1+\gamma_2} \\ & + (w^{r+s-1+\gamma_2} + w^r) + (w^{r+s-1+\gamma_2} + w^r) \lesssim H^{r+s-1+\gamma_2} + w^r. \end{aligned} \quad (4.33)$$

Combining (2.40) and (2.41) yields (4.25), (4.26) and (4.27). By the same argument we can prove (4.28), (4.29) and (4.30). From (2.26), we have

$$\lambda^{w,h_1} - \lambda = \frac{a(u^{w,h_1} - u, u^{w,h_1^*} - u^*)}{b(u^{w,h_1}, u^{w,h_1^*})} - \lambda \frac{b(u^{w,h_1} - u, u^{w,h_1^*} - u^*)}{b(u^{w,h_1}, u^{w,h_1^*})}. \quad (4.34)$$

Note that  $u_H$  and  $u^{w,h_1}$  just approximate the same eigenfunction  $u$ ,  $u_H^*$  and  $u^{w,h_1^*}$  approximate the same adjoint eigenfunction  $u^*$ ,  $|b(u_H, u_H^*)|$  has a positive lower

bound uniformly with respect to  $H$ , thus  $b(u^{w,h_1}, u^{w,h_1*})$  has a positive lower bound uniformly. Combining (4.25), (4.26), (4.28), (4.29) and (4.34) yields (4.31).  $\square$

For convenient argument, we assume  $s_2 = s$ ,  $\gamma_1 = \gamma_2 = \gamma$  in the following Theorem.

**Theorem 4.4.** Under the conditions of Theorem 4.1, we further assume that  $R(\Omega_i)$  holds ( $i = 1, 2, \dots, l$ ), and

$$w^r = \mathcal{O}(H^{r+s-1+\gamma}), \quad h_l^{r+s-1} \gtrsim H^{r+s-1+\gamma}. \quad (4.35)$$

Then there exists  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$  such that

$$\|u^{w,h_l} - u\|_{1,\Omega} \lesssim h_l^{r+s-1}, \quad (4.36)$$

$$\|u^{w,h_l} - u\|_{0,\Omega} \lesssim H^{r+s-1+\gamma}, \quad (4.37)$$

$$\|u^{w,h_l} - u\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma}, \quad (4.38)$$

$$\|u^{w,h_l*} - u^*\|_{1,\Omega} \lesssim h_l^{r+s-1}, \quad (4.39)$$

$$\|u^{w,h_l*} - u^*\|_{0,\Omega} \lesssim H^{r+s-1+\gamma}, \quad (4.40)$$

$$\|u^{w,h_l*} - u^*\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma}, \quad (4.41)$$

$$|\lambda^{w,h_l} - \lambda| \lesssim h_l^{2r+2s-2}. \quad (4.42)$$

**Proof.** Let  $u \in M(\lambda)$  and  $u^* \in M^*(\lambda^*)$ , such that  $u - u_H$  and  $u^* - u_H^*$  both satisfy Lemma 2.2. The proof of (4.36)-(4.42) is completed by induction. When  $l = 1$ , Scheme 3.2 is actually Scheme 3.1. Hence, from Theorem 4.1, Theorem 4.3 and (4.35) we know that (4.36)-(4.42) hold for  $l = 0, 1$ . Suppose (4.36)-(4.42) hold for  $l-2, l-1$ , i.e.,

$$\|u^{w,h_{l-2}} - u\|_{1,\Omega} \lesssim h_{l-2}^{r+s-1}, \quad (4.43)$$

$$\|u^{w,h_{l-2}} - u\|_{0,\Omega} \lesssim H^{r+s-1+\gamma}, \quad (4.44)$$

$$\|u^{w,h_{l-2}} - u\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma}, \quad (4.45)$$

$$\|u^{w,h_{l-2}*} - u^*\|_{1,\Omega} \lesssim h_{l-2}^{r+s-1}, \quad (4.46)$$

$$\|u^{w,h_{l-2}*} - u^*\|_{0,\Omega} \lesssim H^{r+s-1+\gamma}, \quad (4.47)$$

$$\|u^{w,h_{l-2}*} - u^*\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma}, \quad (4.48)$$

$$|\lambda^{w,h_{l-2}} - \lambda| \lesssim h_{l-2}^{2r+2s-2}; \quad (4.49)$$

and

$$\|u^{w,h_{l-1}} - u\|_{1,\Omega} \lesssim h_{l-1}^{r+s-1}, \quad (4.50)$$

$$\|u^{w,h_{l-1}} - u\|_{0,\Omega} \lesssim H^{r+s-1+\gamma}, \quad (4.51)$$

$$\|u^{w,h_{l-1}} - u\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma}, \quad (4.52)$$

$$\|u^{w,h_{l-1}*} - u^*\|_{1,\Omega} \lesssim h_{l-1}^{r+s-1}, \quad (4.53)$$

$$\|u^{w,h_{l-1}*} - u^*\|_{0,\Omega} \lesssim H^{r+s-1+\gamma}, \quad (4.54)$$

$$\|u^{w,h_{l-1}*} - u^*\|_{1,\Omega \setminus \overline{F}} \lesssim H^{r+s-1+\gamma}, \quad (4.55)$$

$$|\lambda^{w,h_{l-1}} - \lambda| \lesssim h_{l-1}^{2r+2s-2}. \quad (4.56)$$

Next we shall prove that (4.36)-(4.42) hold for  $l$ . Using the above formula and Lemma 2.4 to estimate the terms at the right hand side of (4.9) gives

$$\begin{aligned} \|u^{w,h_l} - P_{h_l}u\|_{1,\Omega} &\lesssim h_l^{r+s-1+\gamma} + h_{l-1}^\gamma (h_l^{r+s-1} + h_{l-1}^{r+s-1}) + H^{r+s-1+\gamma} \\ &\quad + H^{r+s-1+\gamma} + (H^{r+s-1+\gamma} + h_l^r) + H^{r+s-1+\gamma} \lesssim H^{r+s-1+\gamma}. \end{aligned} \quad (4.57)$$

Combining (2.40), (2.41) and (4.57) yields (4.36), (4.37) and (4.38). By the same argument we can prove (4.39), (4.40) and (4.41). From (2.26), we have

$$\lambda^{w,h_l} - \lambda = \frac{a(u^{w,h_l} - u, u^{w,h_l*} - u^*)}{b(u^{w,h_l}, u^{w,h_l*})} - \lambda \frac{b(u^{w,h_l} - u, u^{w,h_l*} - u^*)}{b(u^{w,h_l}, u^{w,h_l*})}. \quad (4.58)$$

Using the similar argument as that of Theorem 4.3 we know that  $b(u^{w,h_l}, u^{w,h_l*})$  has a positive lower bound uniformly. Combing (4.36), (4.37), (4.39), (4.40) and (4.58) yields (4.42).  $\square$

**Remark 4.1.**  $\bar{\Omega}_1$  in (3.7) and (3.9) can be different from that in (3.6) and (3.8), which also ensure that the corresponding estimates in Theorem 4.2- Theorem 4.4 still hold.

**Remark 4.2.** By dropping the steps computing  $u_H^*, u^{w*}, e^{h*}, u^{w,h_i*}, \lambda^{w,h_i*}$  in Scheme 3.2, and by replacing  $\lambda^w = \frac{a(u^w, u^{w*})}{b(u^w, u^{w*})}$  and  $\lambda^{w,h_i*} = \frac{a(u^{w,h_i}, u^{w,h_i*})}{a(u^{w,h_i}, u^{w,h_i*})}$  with  $\lambda^w = \frac{a(u^w, u^w)}{b(u^w, u^w)}$  and  $\lambda^{w,h_i} = \frac{a(u^{w,h_i}, u^{w,h_i})}{a(u^{w,h_i}, u^{w,h_i})}$ , respectively, we are able to establish multilevel discretizations based on local defect-correction for symmetric Eigenvalue Problems. Hence the corresponding estimates in Theorem 4.1-Theorem 4.4 still hold.

**Remark 4.3.** By referring to [8], we can establish the parallel version of Scheme 3.1 and Scheme 3.2 and have the corresponding error estimates in Theorem 4.3-Theorem 4.4 apparently.

## 5 Numerical experiments

### 5.1 Computational method for $(\lambda_H^*, u_H^*)$ (see [24, 25])

Assume that  $(\lambda_H, u_H)$  is obtained from Scheme 3.1 or Step 1 of Scheme 3.2, then  $\lambda_H^* = \bar{\lambda}_H$ , and from [24, 25] we can obtain  $u_H^*$  by using the following approach such that  $|b(u_H, u_H^*)|$  has a positive lower bound uniformly with respect to  $H$ .

Let  $m_0$  be the algebraic multiplicity of  $\lambda_H$  and  $l$  be ascent of  $\lambda_H$ .

Let  $u_N^-$  be the orthogonal projection of  $u_H$  to  $N((\frac{1}{\lambda_H^*} - T_H^*)^l)$ , and  $u_H^* = u_H^- / \|u_H^-\|_0$ . When  $u_H \in N((\frac{1}{\lambda_H^*} - T_H^*)^l)$  it is clear that  $u_H^* = u_H$ . When  $u_H \notin N((\frac{1}{\lambda_H^*} - T_H^*)^l)$ , to find  $u_H^*$ , first we seek a basis  $\{\phi_i\}_1^{m_0}$  of  $N((\frac{1}{\lambda_H^*} - T_H^*)^l)$ , and solve the following equations

$$\sum_{i=1}^{m_0} \alpha_i (\phi_i, \phi_j) = (u_H, \phi_j), \quad j = 1, 2, \dots, m_0, \quad (5.1)$$

then let

$$u_H^- = \sum_{i=1}^{m_0} \alpha_i \phi_i, \quad (5.2)$$

$$u_H^* = u_H^- / \|u_H^-\|_0. \quad (5.3)$$

Obviously,  $u_H^*$  satisfies

$$(\frac{1}{\lambda_H^*} - T_H^*)^l u_H^* = 0, \quad \|u_H^*\|_0 = 1.$$



Thus, to find  $u_H^*$  which satisfying (5.2) and (5.3) in  $V_H^0(\Omega)$ , the key is to seek a basis  $\{\phi_i\}_1^{m_0}$  of  $N((\frac{1}{\lambda_H^*} - T_H^*)^l)$ .

When  $l = 1$ , it is actually to solve the following equations to obtain a basis in the solution space.

$$\begin{aligned} u_H^{(1)} &\in V_H^0(\Omega), \\ a(v, u_H^{(1)}) - \lambda_H b(v, u_H^{(1)}) &= 0, \quad \forall v \in V_H^0(\Omega). \end{aligned} \quad (5.4)$$

(When  $\lambda_H$  is a simple eigenvalue,  $l = 1$  and  $N((\frac{1}{\lambda_H^*} - T_H^*)^l)$  is a one-dimensional space spanned by the eigenfunction  $u_h^*$ .)

When  $l > 1$ , how to seek a basis  $\{\phi_i\}_1^{m_0}$  of  $N((\frac{1}{\lambda_H^*} - T_H^*)^l)$  efficiently is an important issue of linear algebra.

## 5.2 Numerical Examples

Consider the convection-diffusion equation

$$-\Delta u + \mathbf{b} \cdot \nabla u = \lambda u, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (5.5)$$

where  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$  or  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$ . The first eigenfunctions of both problems have the singularities at the origin. The exact eigenvalues, which are unknown, are thereby replaced by approximate eigenvalues with high accuracy. For the problem with  $\mathbf{b} = (1, 1)^T$ ,  $\mathbf{b} = (0, 3)^T$  and  $\mathbf{b} = (0, 10)^T$  on  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$ , we take the approximate first eigenvalue as  $\lambda_1 \approx 11.8897$ ,  $\lambda_1 \approx 10.1397$  and  $\lambda_1 \approx 34.6397$ , respectively. For the problem with  $\mathbf{b} = (1, 1)^T$ ,  $\mathbf{b} = (0, 3)^T$  and  $\mathbf{b} = (0, 10)^T$  on  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$ , we take the approximate first eigenvalue as  $\lambda_1 \approx 10.621$ ,  $\lambda_1 \approx 8.871$  and  $\lambda_1 \approx 33.371$ , respectively. We will report some numerical experiments by using linear finite elements on uniform triangle meshes. In our numerical experiments, we use Scheme 3.2 to solve the problem such that  $\Omega_i = (\frac{-1}{2^i}, \frac{1}{2^i})^2 \setminus \{[0, \frac{1}{2^i}] \times [-\frac{1}{2^i}, 0]\}$  for L-shaped domain,  $\Omega_i = (\frac{-1}{2^i}, \frac{1}{2^i})^2 \setminus \{\{0\} \times [-\frac{1}{2^i}, 0]\}$  for slit domain,  $i = 1, 2, \dots, 6$ , and locally fine grids have the same degree of freedom as that of globally mesoscopic grid (see Tables 1-6).

In our experiments, according to the assumptions of Theorem 4.4, we approximately take  $\gamma_1 = \gamma_2 = 1/2$ ,  $2/3$  and  $s = s_2 = 1/2$ ,  $2/3$  so that (4.35) holds for slit domain and L-shaped domain, respectively. We use MATLAB 2011b under the package of Chen (see [6]) to solve the problems, and the numerical results are shown in Tables 1-6. From Tables 1-4 we can see that without increasing degree of freedom on locally fine grids, the first local defect correction can largely improve the accuracy of the eigenvalues, and the local defect corrections that follows can gradually improve the accuracy of the eigenvalues by overcoming the singularity at the origin. But Tables 5-6 also indicate that Scheme 3.2 is not valid for the problems with  $\mathbf{b} = (0, 10)^T$ . Concerning this point, the figures of eigenfunction and its adjoint pair (see Fig. 5.1-5.2) shows their functional-value abrupt changes mainly center on boundary layer, which may lead to the invalidity of Scheme 3.2. for the case  $\mathbf{b} = (0, 10)^T$ , we adopt the parallel version of Scheme 3.2 to make local defect-corrections on boundary layers with functional-value abrupt changes (see also Fig. 5.1-5.2).

Specifically speaking, for the L-shaped domain, we find that it's better to make local defect-corrections near the origin on slightly small area  $\Omega_i^1 = (\frac{-1}{2^{i+1}}, \frac{1}{2^{i+1}})^2 \setminus \{\{0\} \times [-\frac{1}{2^{i+1}}, 0]\}$  for both eigenfunction and its adjoint eigenfunction; as for the other local defect-correction areas, we set as  $\Omega_i^2 = (\frac{-1}{2^{i-1}}, \frac{1}{2^{i-1}}) \times$

$(1 - \frac{1}{2^i}, 1)$  for the eigenfunction,  $\Omega_i^3 = (-\frac{1}{2} - \frac{1}{2^i}, -\frac{1}{2} + \frac{1}{2^i}) \times (-1, -1 + \frac{1}{2^i})$  for the adjoint eigenfunction, respectively; the related numerical results is given in Table 7. Here we set

$$DOF_w \approx \frac{3}{4} \times DOF_{\Omega_i^1} \approx \frac{3}{2} \times DOF_{\Omega_i^2} \approx 4 \times DOF_{\Omega_i^3} \quad (i = 1, 2, \dots).$$

For the slit domain, we set as the local defect-correction area  $\Omega_i^1 = (\frac{-1}{2^i}, \frac{1}{2^i})^2 \setminus \{\{0\} \times [-\frac{1}{2^i}, 0]\}$  for both eigenfunction and its adjoint eigenfunction,  $\Omega_i^2 = (\frac{-1}{2^{i-1}}, \frac{1}{2^{i-1}}) \times (1 - \frac{1}{2^i}, 1)$  for the eigenfunction,  $\Omega_i^3 = (\frac{1}{2} - \frac{1}{2^i}, \frac{1}{2} + \frac{1}{2^i}) \times (-1, -1 + \frac{1}{2^i})$  and  $\Omega_i^4 = (-\frac{1}{2} - \frac{1}{2^i}, -\frac{1}{2} + \frac{1}{2^i}) \times (-1, -1 + \frac{1}{2^i})$  for the adjoint eigenfunction, respectively; the related numerical results is given in Table 8. Here we set

$$DOF_w = DOF_{\Omega_i^1} = DOF_{\Omega_i^2} \approx 2 \times DOF_{\Omega_i^3} = 2 \times DOF_{\Omega_i^4} \quad (i = 1, 2, \dots).$$

Table 7.12 and Table 7.16 in [5] show that, using the adaptive homotopy method to solve the L-shaped domain problem with  $\mathbf{b} = (10, 0)^T$ , the approximate eigenvalue can have 4-5 significant digits with  $DOF = 154994$  and 124469, thus the adaptive homotopy method is efficient. However, by using our algorithm, the approximate eigenvalue can have 6 significant digits with  $DOF_H = 12033$  (see Table 7), which also indicates our algorithm is efficient.

**Table 1:**  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$ ,  $b = (0, 3)^T$ .

| $DOF_H$            | $DOF_w$            | $\lambda_H$        | $\lambda^w$        | $\lambda^{w, h_1}$ |
|--------------------|--------------------|--------------------|--------------------|--------------------|
| 705                | 2945               | 11.94916           | 11.91247           | 11.89949           |
| 2945               | 12033              | 11.91250           | 11.89859           | 11.89343           |
| 12033              | 195585             | 11.89859           | 11.89109           | 11.89028           |
| $\lambda^{w, h_2}$ | $\lambda^{w, h_3}$ | $\lambda^{w, h_4}$ | $\lambda^{w, h_5}$ | $\lambda^{w, h_6}$ |
| 11.89466           | 11.89275           | -                  | -                  | -                  |
| 11.89146           | 11.89068           | 11.89037           | -                  | -                  |
| 11.88996           | 11.88983           | 11.88978           | -                  | -                  |

**Table 2:**  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$ ,  $b = (1, 1)^T$ .

| $DOF_H$            | $DOF_w$            | $\lambda_H$        | $\lambda^w$        | $\lambda^{w, h_1}$ |
|--------------------|--------------------|--------------------|--------------------|--------------------|
| 705                | 2945               | 10.21836           | 10.16730           | 10.15332           |
| 2945               | 12033              | 10.16730           | 10.14979           | 10.14438           |
| 12033              | 195585             | 10.14979           | 10.14117           | 10.14034           |
| $\lambda^{w, h_2}$ | $\lambda^{w, h_3}$ | $\lambda^{w, h_4}$ | $\lambda^{w, h_5}$ | $\lambda^{w, h_6}$ |
| 10.14836           | 10.14643           | -                  | -                  | -                  |
| 10.14238           | 10.14160           | 10.14129           | -                  | -                  |
| 10.14002           | 10.13989           | 10.13984           | -                  | -                  |

**Table 3:**  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$ ,  $b = (0, 3)^T$ .

| $DOF_H$           | $DOF_w$           | $\lambda_H$       | $\lambda^w$       | $\lambda^{w,h_1}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 945               | 3937              | 10.79397          | 10.70640          | 10.66393          |
| 3937              | 16065             | 10.70630          | 10.66356          | 10.64247          |
| 16065             | 64897             | 10.66353          | 10.64237          | 10.63186          |
| $\lambda^{w,h_2}$ | $\lambda^{w,h_3}$ | $\lambda^{w,h_4}$ | $\lambda^{w,h_5}$ | $\lambda^{w,h_6}$ |
| 10.64333          | 10.63315          | -                 | -                 | -                 |
| 10.63208          | 10.62691          | 10.62434          | -                 | -                 |
| 10.62664          | 10.62404          | 10.62274          | 10.62209          | -                 |

**Table 4:**  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$ ,  $b = (1, 1)^T$ .

| $DOF_H$           | $DOF_w$           | $\lambda_H$       | $\lambda^w$       | $\lambda^{w,h_1}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 945               | 3937              | 9.06244           | 8.96099           | 8.91741           |
| 3937              | 16065             | 8.96090           | 8.91470           | 8.89333           |
| 16065             | 64897             | 8.91468           | 8.89266           | 8.88207           |
| $\lambda^{w,h_2}$ | $\lambda^{w,h_3}$ | $\lambda^{w,h_4}$ | $\lambda^{w,h_5}$ | $\lambda^{w,h_6}$ |
| 8.89663           | 8.88641           | -                 | -                 | -                 |
| 8.88289           | 8.87772           | 8.87514           | -                 | -                 |
| 8.87684           | 8.87424           | 8.87294           | 8.87229           | -                 |

**Table 5:**  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$ ,  $b = (0, 10)^T$ .

| $DOF_H$           | $DOF_w$           | $\lambda_H$       | $\lambda^w$       | $\lambda^{w,h_1}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 705               | 2945              | 34.58756          | 34.63484          | 34.61999          |
| 2945              | 12033             | 34.63473          | 34.64168          | 34.63605          |
| 12033             | 195585            | 34.64167          | 34.64066          | 34.63982          |
| $\lambda^{w,h_2}$ | $\lambda^{w,h_3}$ | $\lambda^{w,h_4}$ | $\lambda^{w,h_5}$ | $\lambda^{w,h_6}$ |
| 34.61632          | 34.61460          | -                 | -                 | -                 |
| 34.63437          | 34.63363          | 34.63333          | -                 | -                 |
| 34.63952          | 34.63939          | 34.63934          | -                 | -                 |

**Table 6:**  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$ ,  $b = (0, 10)^T$ .

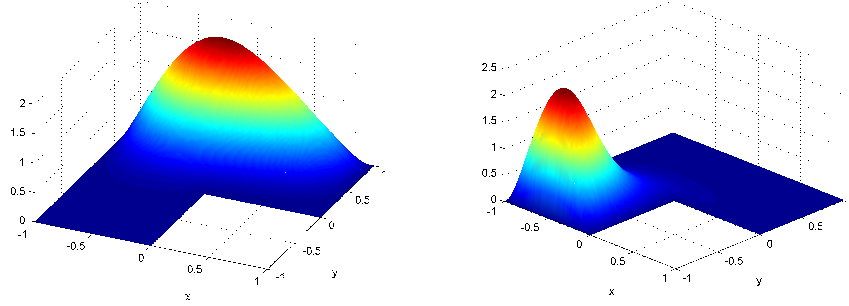
| $DOF_H$           | $DOF_w$           | $\lambda_H$       | $\lambda^w$       | $\lambda^{w,h_1}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 945               | 3937              | 33.43287          | 33.42950          | 33.38763          |
| 3937              | 16065             | 33.42885          | 33.40686          | 33.38587          |
| 16065             | 64897             | 33.40671          | 33.39070          | 33.38021          |
| $\lambda^{w,h_2}$ | $\lambda^{w,h_3}$ | $\lambda^{w,h_4}$ | $\lambda^{w,h_5}$ | $\lambda^{w,h_6}$ |
| 33.36894          | 33.35923          | -                 | -                 | -                 |
| 33.37595          | 33.37090          | 33.36836          | -                 | -                 |
| 33.37510          | 33.37253          | 33.37124          | 33.3706           | -                 |

**Table 7:**  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$ ,  $b = (0, 10)^T$ .

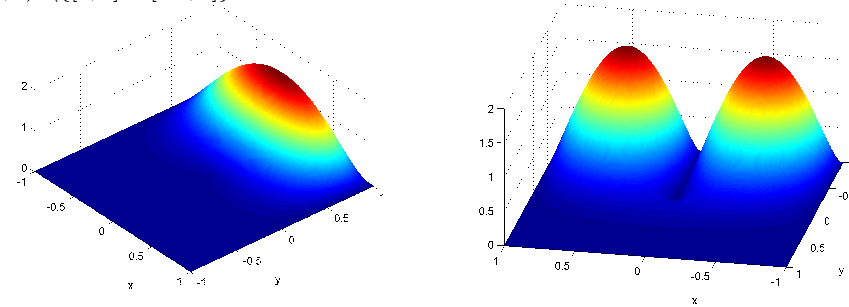
| $DOF_H$           | $DOF_w$           | $\lambda_H$       | $\lambda^w$       | $\lambda^{w,h_1}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 705               | 2945              | 34.58756          | 34.63484          | 34.64245          |
| 2945              | 12033             | 34.63474          | 34.64169          | 34.64162          |
| 12033             | 195585            | 34.64166          | 34.64067          | 34.64017          |
| $\lambda^{w,h_2}$ | $\lambda^{w,h_3}$ | $\lambda^{w,h_4}$ | $\lambda^{w,h_5}$ | $\lambda^{w,h_6}$ |
| 34.64102          | 34.63949          | -                 | -                 | -                 |
| 34.64049          | 34.63980          | 34.63951          | -                 | -                 |
| 34.63990          | 34.63978          | 34.63973          | -                 | -                 |

**Table 8:**  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$ ,  $b = (0, 10)^T$ .

| $DOF_H$           | $DOF_w$           | $\lambda_H$       | $\lambda^w$       | $\lambda^{w,h_1}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 945               | 3937              | 33.43287          | 33.42950          | 33.40090          |
| 3937              | 16065             | 33.42885          | 33.40686          | 33.38916          |
| 16065             | 64897             | 33.40671          | 33.39070          | 33.38103          |
| $\lambda^{w,h_2}$ | $\lambda^{w,h_3}$ | $\lambda^{w,h_4}$ | $\lambda^{w,h_5}$ | $\lambda^{w,h_6}$ |
| 33.38429          | 33.37474          | -                 | -                 | -                 |
| 33.37976          | 33.37475          | 33.37221          | -                 | -                 |
| 33.37605          | 33.37349          | 33.37220          | 33.37155          | -                 |



**Fig. 5.1.** Eigenfunfuntion and its adjoint pair with  $b = (0, 10)^T$  on  $\Omega = (-1, 1)^2 \setminus \{[0, 1] \times [-1, 0]\}$



**Fig. 5.2.** Eigenfunfuntion and its adjoint pair with  $b = (0, 10)^T$  on  $\Omega = (-1, 1)^2 \setminus \{\{0\} \times [-1, 0]\}$

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